Chaotic Coupled Map Lattices

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1 Introduction

When a system of chaotic maps is coupled in a way that allows them to share information about each other's state, there is a possibility that they may synchronize. Each individual map may still be chaotic but has exactly the same state as its synchronized partner. An entire network of maps may be arranged in such a manner as to allow for not only the possible synchronization of all members, but also more complicated modes of quasi-synchronization, where each member is synchronized with at least one partner but not necessarily all partners. Some possible methods of coupling include diffusive (or nearest neighbor) coupling, and one-way coupling, but it is easy to conceive of much more complicated networks. One-way coupling involves choosing a direction along the lattice for each member to communicate, an anology is the game Telephone where a person whispers a message to the person on his right and the message is transmitted down a line of people until the last person tells it to the person who created the message, usually sounding little if anything like the original message (in which case the network would be closed, but this need not be the case). Diffusive coupling is the same concept except that instead of going one way, the message is told to everyone in the immediate vicinity. Think of breaking news spreading in a tightly packed crowd. In the case of these networks we may control the coupling strength, which is something like how tightly packed the crowd is (without changing the number of people). It is interesting to look at the behavior of the system as this coupling parameter changes.

2 Theory

So far we have studied the circle lattice of logistic maps under diffusive coupling. The map corresponding to this can be written as

$$x_{n+1}^{(i)} = f((1-\epsilon)x_n^{(i)} + \frac{\epsilon}{2}(x_n^{(i+1)} + x_n^{(i-1)}))$$

which says that the $n+1$ state of the ith location on the lattice is a weighted mean of its own state and the $i+1$ and the $i-1$ locations, where ϵ is the coupling parameter. In our case the function is the logistic map

$$x_{n+1} = ax_n(1-x_n)$$

We may also represent the lattice as a vector where each dimension is a lattice member:

$$\vec{x}_{n+1} = F((1-\epsilon)I\vec{x}_n + \frac{\epsilon}{2}(R\vec{x}_n + L\vec{x}_n))$$

where R and L are shift operators (corresponding respectively to Up and Down shifts on the vector, which are Right and Left shifts on the lattice), I the identity matrix, and it is understood that the function F acts on the vector elementwise.

Then we may represent the coupling as one matrix, C, which acts on the vector, $\vec{x}_n$.

$$C = (1-\epsilon)I + \frac{\epsilon}{2}(R+L)$$

So that in the case of a diffusively coupled circle lattice with 4 members

$$C = \begin{pmatrix} (1-\epsilon) & \frac{\epsilon}{2} & 0 & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & (1-\epsilon) & \frac{\epsilon}{2} & 0 \\ 0 & \frac{\epsilon}{2} & (1-\epsilon) & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & 0 & \frac{\epsilon}{2} & (1-\epsilon) \end{pmatrix}$$

This type of matrix is called a *circulant matrix* and it may be diagonalized by performing a discrete fourier transform (DFT) on the matrix. This will be convenient in the derivation of the synchronization condition given below.

Conditions for Stable Synchronization

If we consider a phase space with each dimension corresponding to one lattice member's state, then there is a subspace which corresponds to synchronization: the diagonal $x^{(1)} = x^{(2)} = \ldots = x^{(m)}$. We wish to consider the behavior of the system in the vicinity of the diagonal. Do trajectories on the diagonal stay on the diagonal? Of course, since we are dealing with copies of the same deterministic map, a trajectory on the diagonal just corresponds to the simulatenous iteration of the same map with the exact same values, and there is nothing to knock the system out of synchronization. Is the diagonal linearly stable, i.e do trajectories close to the diagonal end up on the diagonal? That is a much more complicated question. Rewording it slightly we can ask: for what values of the coupling parameter, ϵ does the iteration of the lattice contract every direction in phase space except along the diagonal? Singular values give us a way to look at the directions which an operator stretches or contracts. For example if a matrix is applied to the unit circle transforming it into an arbitrary ellipse, the largest singular value is length of the semi-major axis, and the second singular value is the length of the semi-minor axis. So to see if the diagonal is linearly stable we can look at the singular values of the Jacobian, which are just the absolute value of the eigenvalues in decreasing order. In our case we can analytically approximate the singular values in the vicinity of the diagonal.

The jacobian for a circle lattice looks like $J = f'(\bar{x})C$ [1], where C is the circulant matrix and is a $f'(\bar{x})$ is the derivative of the mean of entries of the state vector $\bar{x}$. The eigenvectors of C are the columns of the DFT matrix which have the form:

$$\vec{w_k} = \begin{pmatrix} 1 \\ \zeta_k \\ \zeta_k^2 \\ \vdots \\ \zeta_k^{m-1} \end{pmatrix}$$

where m is the size of the lattice $\zeta_k = e^{\frac{2\pi i k}{m}}$. Applying J to this vector

$$J\vec{w_k} = f'(y)\left[(1-\epsilon)\vec{w_k} + \frac{\epsilon}{2}(R\vec{w_k} + L\vec{w_k})\right]$$

[1] the reason we consider the mean is addressed in the notes

But shifting $\vec{w}$ right or left is the same as multiplying by ζ_k and ζ_k^{-1} respectively.

$$J\vec{w_k} = f'(y)\left[(1-\epsilon) + \frac{\epsilon}{2}(\zeta_k + \zeta_k^{-1})\right]\vec{w_k}$$
$$J\vec{w_k} = f'(y)\left[(1-\epsilon) + \frac{\epsilon}{2}(e^{\frac{2\pi ki}{m}} + e^{\frac{-2\pi ki}{m}})\right]\vec{w_k}$$
$$J\vec{w_k} = f'(y)\left[(1-\epsilon) + \epsilon\left(\cos\left(\frac{2\pi k}{m}\right)\right)\right]\vec{w_k}$$

Having found an expression for the eigenvalues, we see that the singular values, s_k, have the form

$$s_k = \left|f'(y)\left[(1-\epsilon) + \epsilon(\cos\left(\frac{2\pi k}{m}\right)\right]\right|$$

A sufficient condition for linear stability would be if for all iterations, all the singular values except to the one corresponding to stretching along the diagonal are less then one.

In the case of the logistic map we may bound the derivative by the logistic parameter $|f'(y)| < a$. So the sufficient condition for synchronization becomes

$$a\left|1 + \epsilon\left(\cos\left(\frac{2\pi k}{m}\right) - 1\right)\right| < 1$$
$$\left|1 + \epsilon\left(\cos\left(\frac{2\pi k}{m}\right) - 1\right)\right| < 1/a$$
$$-1/a < 1 + \epsilon\left(\cos\left(\frac{2\pi k}{m}\right) - 1\right) < 1/a$$
$$-(1+1/a) < \epsilon\left(\cos\left(\frac{2\pi k}{m}\right) - 1\right) < 1/a - 1$$
$$\frac{1-1/a}{1-\cos\left(\frac{2\pi k}{m}\right)} < \epsilon < \frac{1+1/a}{1-\cos\left(\frac{2\pi k}{m}\right)} \tag{1}$$

However this is a very weak condition in the sense that though it guaruntees synchronization, it only applies to a small subset of what is observed to actually synchronize. The maximum value of a for which equation (1) may be satisfied is $a = 3$ which in the case of the logistic map is not even large enough to be chaotic, though the condition is irrespective of the exact map. With that in mind the following analysis provides more realistic bounds.

Stronger Condition for Synchronization

We must first describe a synchronized solution [2], $\vec{y} = f(\bar{x}_n)\vec{\mathbf{1}}$, where $\vec{\mathbf{1}}$ represents a vector of 1's. The difference between the actual state of the lattice and the synchronized state is $\vec{\delta_n} = \vec{x}_n - \vec{y}_n$ so that $\vec{\delta}_{n+1} = F(C\vec{x}_n) - \vec{y}_{n+1}$ It is convenient to rewrite this equation as

$$\vec{y}_{n+1} + \vec{\delta}_{n+1} = F(\vec{y}_n + C\vec{\delta}_n)$$

[2] see Notes

By substituting $\vec{y}_n + C\vec{\delta}_n = \vec{x}_n$ so that the argument of F becomes

$$(1-\epsilon)(\vec{y}_n + \vec{\delta}_n) + \left[(1-\epsilon)I + \frac{\epsilon}{2}(R+L)\right](\vec{y}_n + \vec{\delta}_n =$$
$$(1-\epsilon+\epsilon)\vec{y}_n + \left[(1-\epsilon)I + \frac{\epsilon}{2}(R+L)\right]\vec{\delta}_n =$$
$$\vec{y}_n + C\vec{\delta}_n$$

Recall that $\vec{y}_n$ is a constant vector and therefore unaffected by the shift operators. We linearize around synchronization where $\vec{\delta}_n = 0$

$$\vec{y}_{n+1} + \vec{\delta}_{n+1} \approx F(\vec{y}_n) + F'(\vec{y}_n)\vec{\delta}_n + \ldots$$

where $F'(\vec{y}_n)$ is the Jacobian. Since $\vec{y}_{n+1} = F(\vec{y}_n)$, we may cancel the two leaving

$$\vec{\delta}_{n+1} \approx F'(\vec{y}_n)C\vec{\delta}_n$$

Now as the map $\vec{\delta}_n$ is iterated N times the factor $F'(\vec{y}_n)C$ becomes the product $\prod_{j=1}^{N} F'(y_j)C^j$, and we wish to see if on average $\vec{\delta}_{n+1}$ is growing or shrinking. We know how to diagonalize C, call the diagonalized matrix D. So for N iterations we take the Nth root, i.e. the geometric mean

$$\left[\prod_{j=1}^{N} F'(y_j)D^j\right]^{1/N}$$

$$D\left[\prod_{j=1}^{N} F'(y_j)\right]^{1/N}$$

In the limit as $N \to \infty$, the geometric mean of the product of the derivatives just becomes the lyapunov number for the logistic map, e^{λ}. This gives the equation

$$\vec{\delta}_{n+1} \approx e^{\lambda} D\vec{\delta}_n$$

We know the eigenvalues of D and this gives us an equation for the singular values

$$s_k = e^{\lambda}\left|1 - \epsilon\left(\cos\left(\frac{2\pi k}{m}\right)\right)\right|$$

where $k = 0, 1, \ldots, m-1$. For synchronization we need all the singular values except s_0 to be less than one.

$$e^{\lambda}\left|(1-\epsilon) + \epsilon\cos\left(\frac{2\pi k}{m}\right)\right| < 1$$

Which gives the following condition for synchronization

$$\left|1 + \epsilon\left(\cos\left(\frac{2\pi k}{m}\right) - 1\right)\right| < e^{-\lambda}$$

The different modes correspond to frequencies of quasi-synchronization along the lattice. It is an observation that the total synchronization mode has never been blocked out, but another mode may become more prominent.

For the 3-lattice we get the condition

$$\frac{2}{3}(1-e^{-\lambda}) < \epsilon < \frac{2}{3}(1+e^{-\lambda})$$

For maps larger than $m = 3$ the expression can be refined by first letting $k = 1$ and then $k = m/2$ if the lattice-size is even or $k = (m \pm 1)/2$ if it is odd. These are key modes to supress. For a general even size lattice we find

$$\frac{1-e^{-\lambda}}{1-\cos(\frac{2\pi}{m})} < \epsilon < \frac{1+e^{-\lambda}}{2}$$

for which

$$\lambda_{max} = \ln \left| \frac{3-\cos\left(\frac{2\pi}{m}\right)}{1+\cos\left(\frac{2\pi}{m}\right)} \right|$$

The condition for odd lattice size just substitutes $m \pm 1$ for m. An illustrative example is the case of the 4-lattice, where we find that

$$|1-\epsilon| < e^{-\lambda}$$

and

$$|1-2\epsilon| < e^{-\lambda}$$

respectively. Solving for ϵ we get

$$1-e^{-\lambda} < \epsilon < 1+e^{-\lambda}$$

$$\frac{1-e^{-\lambda}}{2} < \epsilon < \frac{1+e^{-\lambda}}{2}$$

Taking the strictest conditions from both equations we have

$$1-e^{-\lambda} < \epsilon < \frac{1+e^{-\lambda}}{2}$$

From this expression we may solve for the maximum value of λ for which a lattice, $m \geq 4$, may reliably synchronize: $\lambda_{max} = \ln 3$.

This does not apply to the 3-lattice which may synchronize for all λ, however the range of ϵ decays exponentially with respect to increasing λ. If r is the length of the interval of ϵ then $r = \frac{4}{3}e^{-\lambda}$. Intuitively, it makes sense that if a map is less chaotic, having a smaller λ, then it should be easier to synchronize. Indeed for larger maps, synchronization may only be stable for very small λ. It is important to make explicit what this condition actually says. All the condition can say is that if synchronization happens and ϵ satisfies these conditions then synchronization will be stable. However the data seems to relate very well to these bounds with the noted exception of the circle map, which means that there is still more to be understood. There could be a fault in the program or an unforseen phenomenon.

3 Data

Matlab was used to model a few lattices for a wide range of parameters. The logistic map and a torus map

$$x_{n+1} = ax_n \mod 1$$

was used to test the synchronization conditions on the 4-lattice. In order to test for synchronization the standard deviation, σ, was taken of all the lattice points and then averaged over all iterations. Clearly if this σ_{avg} is zero than the system is fully synchronized. If we were interested in observing other synchronization modes, we can take the discrete fourier transform over all the lattice members. Figure (1) shows a typical situation of a synchronizing lattice, in this case the lattice is a 4-lattice of logistic maps with logistic parameter $a = 3.9$, corresponding to a lyapunov exponent of $\lambda \approx .516$. Notice that in figure (1) there is an

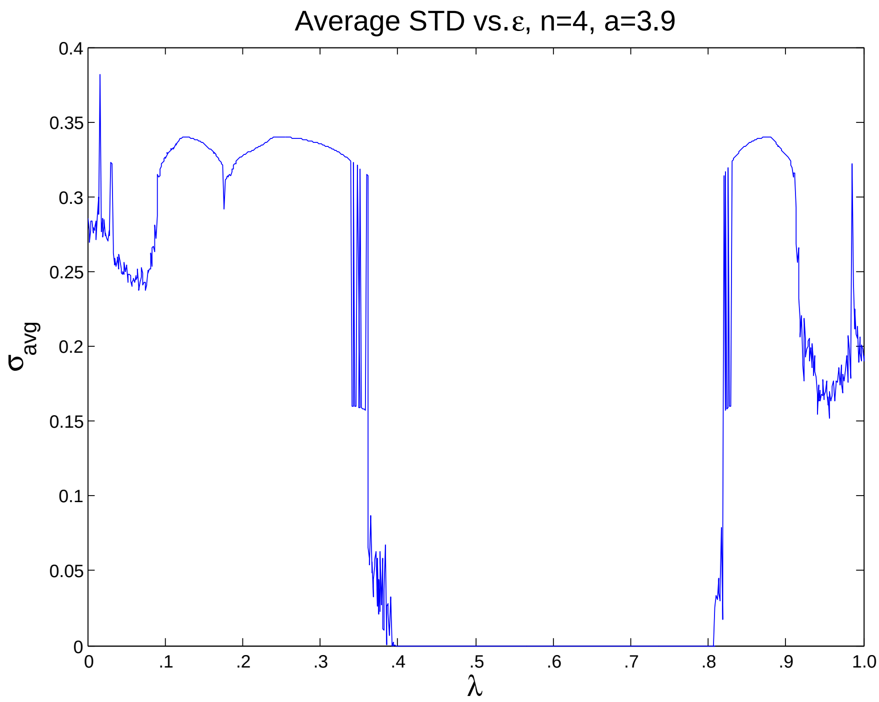

Figure 1: Average Standard Deviation vs. Coupling Parameter for the circle lattice of logistic maps, n=4 a=3.9

interval for which the standard deviation drops to zero. This is the range of the synchronization parameter predicted by the theory to give stable synchronization. Figure (2) shows an example of a lattice which was outside the theoretical bounds for synchronization, the 4-lattice of logistic maps with $a = 4$. In both cases shown there are regions in which the standard deviation forms a smooth curve. If the lattice were completely out of synchronization there would be no correlation at all in this region of the plot. Therefore these probably represent regions of quasi-synchronization and other more complicated behavior such as the emergence of fixed points. If we consider 4-lattices of varying chaoticness we can test the bounds of our theory. Figure (3) shows the synchronizing values of the coupling parameter versus the lyapunov exponent for the logistic and torus maps.

It wasn't until we considered a family of circle maps $x_{n+1} = 2x + a\sin(2\pi x)$

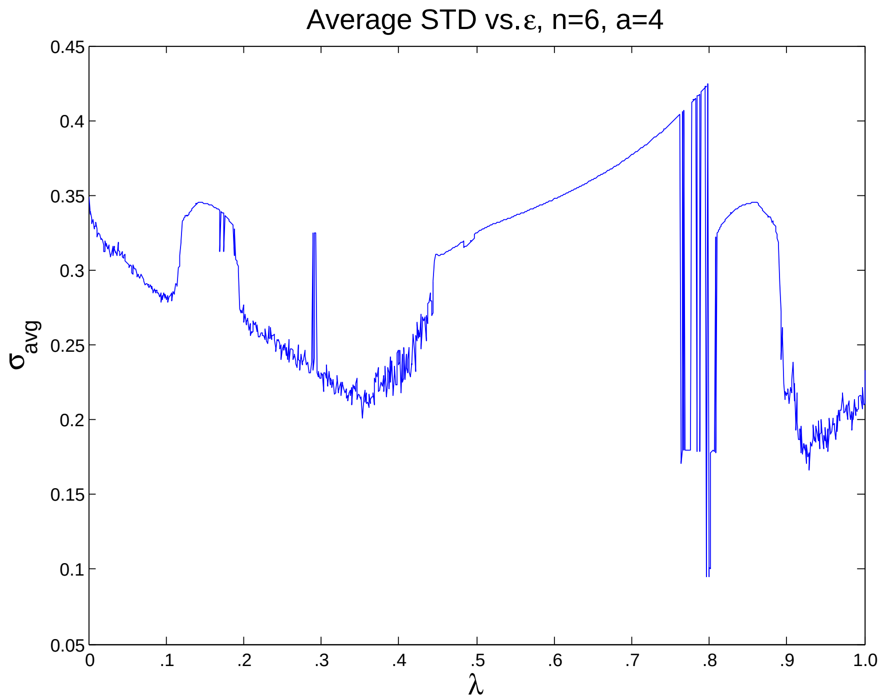

Figure 2: Average Standard Deviation vs. Coupling Parameter for the circle lattice of logistic maps, n=6 a=4

mod 1 that we found a counterexample to the theoretical bounds, shown in figure (4). An interesting observation is that the there is a definite range of synchronizing parameters which fit the theory and then there are outliers which don't. One might be led to conclude that these may be have an average standard deviation of zero but not be linearly stable and therefore the theory still stands, but in fact these were generated with initial conditions slightly perturbed from synchronization so they are linearly stable points.

4 Conclusion

Three different types of maps were looked at, two of which perfectly fit the bounds of synchronization. The third map we looked at had an interval which fit the theory and outliers which didn't. Clearly there is more to be understood here.

5 Future Plans

For the next stage of the project we hope to answer some of the questions which were raised in this report. Furthermore, we will look at the role that symmetry plays in the synchronization of lattices. We have only looked at

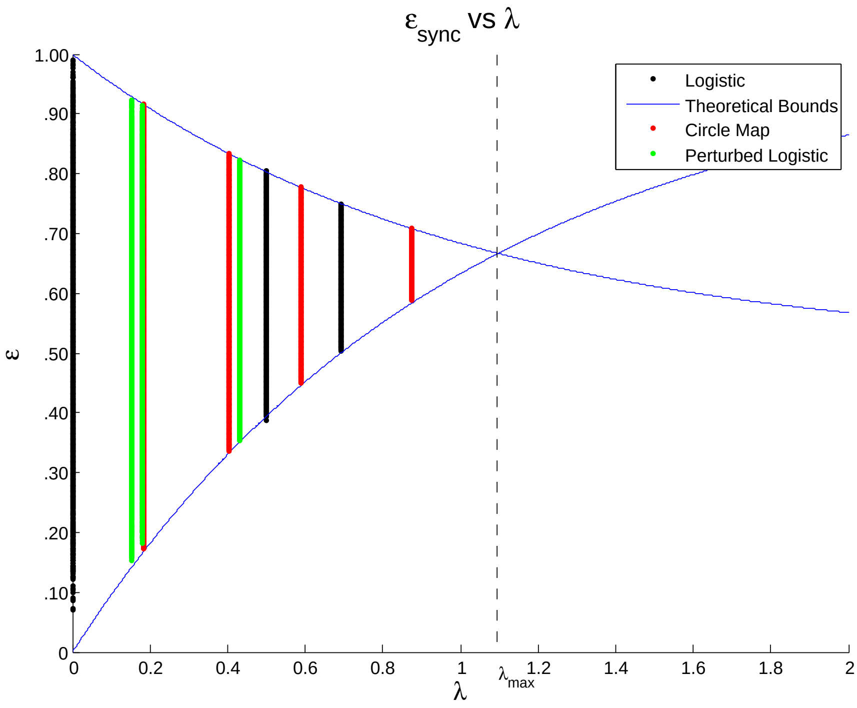

Figure 3: Synchronizing Coupling Parameter Values vs. Lyapunov Exponent

circular lattices under diffusive which have rotational symmetry and this allowed us to diagonalize the jacobian to find explicit equations for the singular values. So it is natural to consider other cases where we can diagonalize the jacobian, and maybe be able to make a general statement about synchronization in such lattices. However, the main focus will be to study the flow of information in general coupled map lattices.

6 Notes

Consider some arbitrary state vector $\vec{x}$, and decompose it into a vector on the diagonal,

$$\vec{y} = \begin{pmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix}$$

and one perpendicular to the diagonal, $\vec{\delta}$, where $\bar{x}$ is the mean of the entries of the state vector. If we consider our lattice map, F, which acts on the state vector $\vec{x}$ giving another vector $F(\vec{x})$ we must know if $F(\vec{x}) - F(\vec{y})$ is also perpendicular to the diagonal. This is because the following analysis depends on the existence of a synchronized solution which is the orbit of $\vec{y}$ and in order for the state

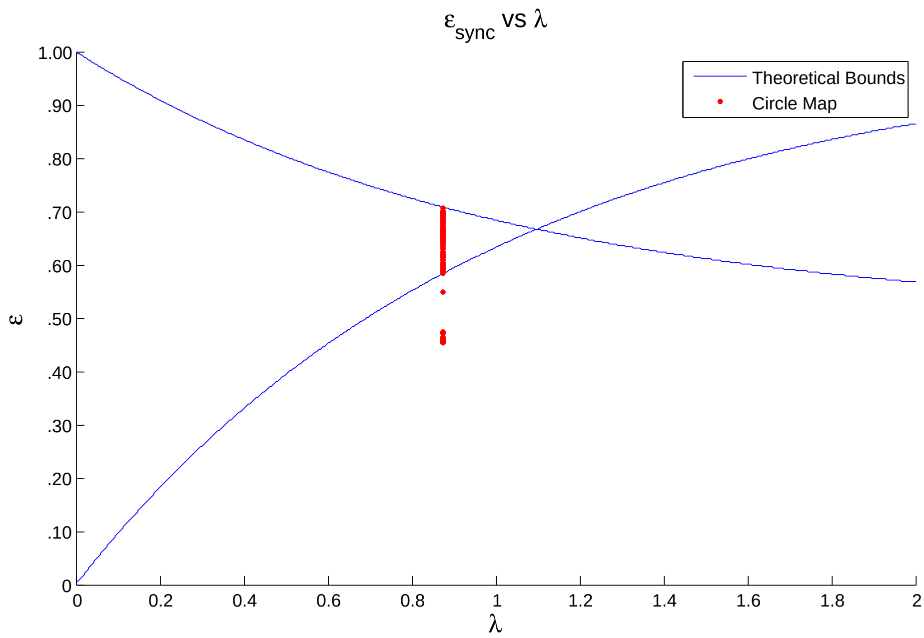

Figure 4: Synchronizing Coupling Parameter Values vs. Lyapunov Exponent for the map $x_{n+1} = 2x + a\sin(2\pi x) \mod 1$ with $a = .16\pi$

solution to converge to the synchronized solution, $\vec{\delta}_{n+1} = F(\vec{x}_n) - F(\vec{y}_n)$ must go to zero. If the singular values in every direction perpendicular to the diagonal are less than 1, and $\vec{\delta}$ is perpendicular to the diagonal then it will go zero. To see that it is, consider $F(\vec{x}) = F(\vec{y} + \vec{\delta})$. Taylor expanding we find:

$$F(\vec{y} + \vec{\delta}) = F(\vec{y}) + F'(\vec{y})\vec{\delta} + \dots$$

which means that

$$F(\vec{y} + \vec{\delta}) - F(\vec{y}) \approx F'(\vec{y})\vec{\delta}$$

And if this is perpendicular to the diagonal then the sum of its elements is zero, given that $\vec{\delta}$ is already perpendicular. Observe

$$F'(\vec{y})\vec{\delta} = f'(\bar{x})C\vec{\delta}$$

$$= f'(\bar{x}) \begin{pmatrix} (1-\epsilon) & \frac{\epsilon}{2} & 0 & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & (1-\epsilon) & \frac{\epsilon}{2} & 0 \\ 0 & \frac{\epsilon}{2} & (1-\epsilon) & \frac{\epsilon}{2} \\ \frac{\epsilon}{2} & 0 & \frac{\epsilon}{2} & (1-\epsilon) \end{pmatrix} \begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} (1-\epsilon)\delta_0 + \frac{\epsilon}{2}\delta_1 + \frac{\epsilon}{2}\delta_3 \\ (1-\epsilon)\delta_1 + \frac{\epsilon}{2}\delta_0 + \frac{\epsilon}{2}\delta_2 \\ (1-\epsilon)\delta_2 + \frac{\epsilon}{2}\delta_1 + \frac{\epsilon}{2}\delta_3 \\ (1-\epsilon)\delta_3 + \frac{\epsilon}{2}\delta_2 + \frac{\epsilon}{2}\delta_0 \end{pmatrix}$$

All ϵ factors cancel out and we are left with $\sum_{k=0}^{3} \delta_k = 0$, which means that the new vector is perpendicular to the diagonal as required. With that in mind the preceding analysis applies.