Lecture #4

Borcherds Product

\[ f = \sum a(n)q^n \in M_{\frac{1}{2}}(\Gamma_0(141)) \rightarrow \text{Mod. form} \]

\[ f_D = \sum a_D(n) q^n \]

D div. \[ H_D(j(w)) \]

q expansion

"Hilbert Class Polynomial"
Given \( D, E \) \( P_D(x) \) factorial fan in \( X \) s.t.

\[
\mathbf{q} \star \prod_{n=1}^{\infty} P_D(q^n)
\]

is a need fan with a "twisted Herges, duum".

\textbf{Note:} Borcherds: \( P_D(x) = 1 - x \)
Example (Ramanujan's mock $G$)

\[ w(q) := \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q; q^2)^{n+1}} \]

\[ \Rightarrow \sum a(n)q^n := -2q^t (w(q^8) + w(-q^8)) \quad \text{if } n \in 2+\frac{1}{2} \]

Fact (Example $D=-8$)

\[ P_{-8}(x) = \frac{1+\sqrt{-2}x-x^2}{1-\sqrt{-2}x-x^2} \]

\[ \Psi(2) := \prod_{n=1}^{\infty} P_{-8}(q^n) \]

\[ \left( \frac{\eta}{\eta} \right) a(n^{1/2}) \]

is a mod functor on \( \Gamma_0(6) \) with a twisted Hecke divisor of div $\mathbb{Z}$.
Fact: For every $D_j$ with 1

$\Rightarrow$ Mf. on $\mathbb{C}_0(b)$ with cy. int. root.

with specific divisor!

Fact: For most $N$, most elts in $\left\{ D_j \right\}_{j \in \mathbb{N}}$ have the property that most of coeff. of $f^*$ are transcendental...
Heegner Points + Heegner Divisors (Special Case)

\[ H_{\frac{1}{2}}(x) \xrightarrow{2_{\frac{1}{2}}} g = \sum b_{E}(a) \chi_{E} \cdot S_{\frac{1}{2}}(a) \]

\[ h = h^{+} + h^{-} \]

What do \( h^{-} + h^{+} \) tell us about \( E \)?

Standard Facts + Definitions

- \( E_{0} \): Quadratic twist of \( E \)

- Kolyvagin's Thm. If \( \text{ord}_{s=1}(L(E, s)) \leq 1 \), then \( \text{rk}(E_{0}/\mathbb{Q}) = \text{ord}_{s=1}(L(E, s)) \)
Theorem. [B-0] THAT:

1) If \( sfe(E_0) = 1 \), then
\[
\chi_h'(0) = \chi_D \{ b_E(1,1) \}.
\]

2) If \( sfe(E_0) = -1 \), then
\[
\chi_h^1(W) \in \mathbb{Z} \iff \chi(\{E_0\}, 1) = 0.
\]

3) If \( sfe(E_0) = 0 \), then \( \chi_h^0(W) \) is trans. \iff \( \chi(\{E_0\}, 1) \neq 0 \).

Fact: Beauville Conjecture

If \( b_E(1,1) \neq 0 \), then \( \text{rk}(E^{1,1}) = 0 \).

By Kiyoshi, we wish to know when \( \chi(E_0, 1) = 0 \).
Idea Behind Proof:

- Gross-Zaslaw \( \rightarrow L' \) to height of Heegaard \( \mathcal{H} \) (Special Heegaard diagram)

- \( h_E = h \left( h^+ \right) e H_2 \)

- Arndt Table for m.s. on \( \mathcal{P} (\chi \mathcal{L}) \) with Heegaard diagram

- Boundary product

- D Dec.

- \( q^k \prod_{n=1}^{\infty} P_D (q^n) \xrightarrow{\text{col}} \text{m.s. a CML} \)

- with twisted Heegaard diagram associated to \( D \)
* Idea: \( L'(\ell_0, 1) = 0 \) when all \( f \binom{\ell^T}{m} \in \mathbb{Q} \).

* Tricks...

\[
\frac{1}{\binom{\ell^T}{m}} \in \mathbb{Q} \implies \frac{\binom{\ell^T}{m}}{\binom{\ell^T}{m}} \in \mathbb{Q}.
\]

* Note: If \( f \in \mathbb{S}_k \) is nontrivial, then by the surjectivity of \( \mathbb{H}_{2-k} \), then it is a "cool"

\[
g \in \mathbb{H}_{2-k} \xrightarrow{\mathbb{H}_{2-k}} f
\]

\[
g + h^f \rightarrow \text{tells us something new.}
\]
Constructions (Standard)

Ramanujan’s Examples

\[ f(q) = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)(1+q^2)(1+q^3) \cdots (1+q^n)} \]

Seem weird!

Question: How are these “strange expression related to recognizable number objects”?

Example:

\[ f(q) = \frac{2}{q^{1/2} \eta(2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^m q^{3n^2/2}}{1+q^n} \]
Auxiliary Factor $\mathcal{A}\rightarrow "Eisenstein\ Series"

Theorem (Zwegers 2002)

If $\gamma\in SL(2,\mathbb{Z})$, $u,v\in \mathbb{C}\setminus (2\pi i \mathbb{Z})$, and define

$$M(u,v;\gamma) = \frac{Z^\frac{1}{2}}{\Theta(v;\gamma)} \cdot \sum_{n\in \mathbb{Z}} \frac{(-u)^n q^n}{1-2q^n}$$

where $Z = e^{2\pi i u}$, $W = e^{2\pi i v}$, $q = e^{2\pi i \tau}$.

$\Theta(v;\gamma) = \text{Jacobi } \theta$-function

Define

$$R(u,v;\gamma) = \text{period integral of a cusp } \frac{3}{2}$$

unary theta function.
Then we have: for \((\phi, \psi) = A \in SL_2(\mathbb{Z}), \) we

\[
\hat{M}\left(\frac{\mu}{8t+8}, \frac{\nu}{8t+8} ; \frac{\eta t + \phi}{8t+8}\right) = (8t+8)^{\frac{1}{2}} \hat{M}(M, \psi)
\]

where \(\hat{M} : = M + \mathbb{R} \).

\[
\hat{M} \rightarrow \text{harmonic Maass form of wt } \frac{1}{2}.
\]
Amazingly, Ramanujan’s guess gives:

<table>
<thead>
<tr>
<th>$q$</th>
<th>-0.990</th>
<th>-0.992</th>
<th>-0.994</th>
<th>-0.996</th>
<th>-0.998</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(q) + b(q)$</td>
<td>3.961...</td>
<td>3.969...</td>
<td>3.976...</td>
<td>3.984...</td>
<td>3.992...</td>
</tr>
</tbody>
</table>

This suggests that

$$\lim_{q \to -1} (f(q) + b(q)) = 4.$$
This suggests that
\[
\lim_{q \to i} (f(q) - b(q)) = 4i.
\]

<table>
<thead>
<tr>
<th>$q$</th>
<th>0.992$i$</th>
<th>0.994$i$</th>
<th>0.996$i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(q)$</td>
<td>$2 \cdot 10^6 - 4.6 \cdot 10^6i$</td>
<td>$2 \cdot 10^8 - 4 \cdot 10^8i$</td>
<td>$1.0 \cdot 10^{12} - 2 \cdot 10^{12}i$</td>
</tr>
<tr>
<td>$f(q) - b(q)$</td>
<td>$\sim 0.05 + 3.85i$</td>
<td>$\sim 0.04 + 3.89i$</td>
<td>$\sim 0.03 + 3.92i$</td>
</tr>
</tbody>
</table>
Crazy Formulas (Ramanujan's Claim)

\[ \lim_{q \to 1} \left( f(q) - (-1)^k b(q) \right) = O(1). \]

- \(2k^{\frac{1}{2}}\) point = root of unity
- with \(\frac{1}{2}\) m.f.
- \(\frac{1}{2}\)

Thm. (F-Or) If \(z\) is \(2k^{\frac{1}{2}}\) primitive root of unity, then

\[ \lim_{q \to 1} \left( f(q) - (-1)^k b(q) \right) = -4 \sum_{n=0}^{k-1} \left( \psi(2n+1, 2k-2n-1) \right) \]

\[ \in \mathbb{Z}[\ell] \]

RHS: \(U(q) \rightarrow \text{"quantum m.f."} \)