2. E elliptic curve \( / \mathbb{Q} \)

\[ P_{E, P} : \mathfrak{G}_E \rightarrow \text{GL}_2(\mathbb{Z}_p) \]

\[ \downarrow \]

\[ \text{GL}_2(\mathbb{Q}_p). \]

\( \{ P_{E, P} \} \) compatible system

\[ E \leftrightarrow f \text{ newform of wt } 2 \]

\[ + \text{ level } \Gamma_0(11) \]

\[ \text{Thm (modularity of elliptic curves)} \]

- bijection
- isogeny classes
  \( \{ \text{E elliptic curve} \} \leftrightarrow \{ \text{normalized newform } f \text{ of wt } 2, \text{ level } \Gamma_0(11), \text{ with integer Fourier coefficients} \} \)
In fact $N = N_E = \text{conductor of } E$

Correspondence: $a_p(f) = a_p(E) (p \nmid N_E)$

Proven by passing through compatible systems of Galois reps.

$f \mapsto \{ P \in \overline{\mathbb{Q}}_p, \beta \}$

[Eichler–Shimura 60s].

Idea: Use the geometric interpretation of $f$ to associate $A_f$, an abelian variety, to $f$.

Tate module of $A_f$ gives a compatible system, and if $f$ has integer coefficients, $A_f$ is an elliptic curve.
Keep assumption that \( f \) has wt 2, forget the assumption on coefficients.

Let \( K_f = \text{coefficient field of } f \).

Then \( G_{K_f} \) acts naturally on \( 1 \)-Type modules of \( A_f \), so if \( \chi \) is a prime of \( A_{K_f} \), get

\[
P_f, \chi : G_{\mathbb{Q}} \rightarrow GL_2(K_{f, \chi}).
\]

Its, unramified outside \( N(1/f) \), and irreducible.

\[
\text{tr}(P_f, \chi)(\mathfrak{f} \mathfrak{b}) = \sigma_e(f) \in N(1/f)
\]

Compatible system.

A single \( P_f, \chi \) is enough to determine \( f \), and so a single \( P_f, \chi \) is enough to determine the whole compatible system.
If $f$ has no $k \neq 2$, then $\sigma = \alpha$. It's still possible to construct the $\mathfrak{P}_{\sigma}$:

- if $k > 2$, Deligne used étale cohomology of Kuga-Sato varieties
- étale cohomology of modular curves with non-constant coefficients.

- $k = 1$, Deligne - Serre used congruence to construct the $\mathfrak{P}_{\sigma}$.

In this course, $k > 2$.

So in general, $f \rightarrow \mathfrak{P}_{\sigma, 1, 3}$.

Want: to go from a compatible system to a modular form.

**Question:** Does every compatible system of eigenforms $\mathfrak{P}_{\sigma}: \mathcal{G}_{\mathfrak{a}} \rightarrow \mathcal{G}_{\mathfrak{L}}(K_{\mathfrak{a}})$ come from a modular form?
\[ K = \text{# field, } \lambda = \text{finite place of } K, \text{ } \mathfrak{p}_{\lambda} \text{ its, mod, unramified outside } N(\mathfrak{m}), \Phi \to \mathfrak{p}_{\lambda}(F_{\lambda}), \mathcal{L} + N(\mathfrak{m}), \text{ independent of } \lambda \]
Fact/easy calculation:
- If $c \in \mathbb{C} \times \mathbb{Q}$ be complex conjugate, then $\det ps, \chi(c) = -1$.

Say that $ps, \chi$ is *odd*.
Maass forms examples all have
$\det p_{s_3}(c) = +1 \quad [p_{s_3} \text{ is even}]

Can avoid Maass forms by saying that $p_{s_3}$ are *odd*.

Conj. If $p_{s_3}$ is a compatible system of odd representations, then $f(x) \in \mathbb{Z}$, $f$ modular form s.t. $R = \mathcal{E}^{-1} \otimes ps, \chi$.

Reasonable conj but probably very hard to prove.
Reason this is hard is that we haven't said anything about $\mathcal{X}_{/\mathbb{Z}_p}$, $p = N\mathbb{Z}$.

"Motto: $\mathcal{X}$ is determined by $\mathcal{X}_{/\mathbb{Z}_p}$".

Idea: $\mathcal{X}_{/\mathbb{Z}_p}$ can be very complicated, and we should try to understand it better.

The way we understand $\mathcal{X}_{/\mathbb{Z}_p}$ is via $p$-adic Hodge theory.

If $f$ has weight $k$, then

$\text{Pr} \times \mathcal{X}_{/\mathbb{Z}_p}$ is de Rham with Hodge-Tate weights $0, k-1$. 
If we believe the Conjecture above, should also believe:

Conj. Let \( \langle px \rangle \) be an odd compatible system of odd mod. reps., with the property that \( \exists \) integers \( a, b \) s.t. \( b > 0 \), and \( \exists \) \( \forall \) for each \( \lambda \), if \( p = M \lambda \), then \( p x \) is de Rham with Hodge-Tate weights \( a, a + b \).

Then \( \exists \) \( \forall \) a modular form of wt \( b + 1 \) s.t. \( p x \otimes \mathfrak{E}^{a} \cong \mathfrak{P}_{\lambda} \).

([Conj \implies \text{Conj}' \text{ using } \mathfrak{E}^{a} \text{ has Hodge-Tate weights } a].)

Advantage of Conj': can actually prove it in a lot of cases.
Conjecture (Fontaine-Mazur).

If \( E/Q_p \) is finite, and

\[
p : \mathbb{G}_m \rightarrow \mathbb{G}_m^2 \quad (E)
\]

is its, odd, irreducible, de Rham at \( p \) [\( \mathbb{G}_m \) is de Rham] unramified at all but finitely many primes.

Then \( \exists a, f \text{ s.t. } p \equiv 3^a \mod \mathfrak{p} \)

for some \( \mathfrak{p} \).

Let \( FM \text{ conj } \Rightarrow \text{ Conj } \)

[Each \( p \) satsifies hypothesis of \( FM \text{ conj } \)].
The Rhs.

- If we drop the Riemann condition, or the condition that \( p \) is unramified a.e., then \( c \) is false.
- This implies that \( p \) is part of a compatible system.
- \( f \) is determined uniquely by \( p \).
- If \( p \) has distinct Hodge-Tate weights, then:
  - \( f \) should have weight \( k > 2 \)
- Conjecturally, 'odd' should follow from the other hypotheses.

[Proved in many cases by FC, using modularity lifting theorems]
[We will keep the assumption of oddness]
Strategy for proving Case 1:
- Choose a "nice" \( \lambda \).
- Prove FM conjecture for \( \overline{\lambda} \).

\[ \overline{\lambda} \text{ is modular} \]

- Prove that \( \overline{\lambda} \), the reduction mod \( p \) of \( \lambda \), is modular [Sene's conjecture].

- Deduce that \( \lambda \) itself is modular [modularity lifting theorems].

\( p : G_A \rightarrow GL_2(E) \) is finite.

Conjugate: \( p : G_A \rightarrow GL_2(G_E) \). Reduce modulo \( m_E \).
\( \overline{p} : G_a \to \text{GL}_2(\mathbb{F}) \quad \mathbb{F} = \mathbb{F}_q \text{/ \text{finite}}. \)

This is only well-defined up to semi-simplification.
Assume: \( \overline{p} \) is absolutely irreducible.
Then \( \overline{p} \) is well-defined, depends only on \( p \).

Scone\'s conject [mod-p version of FM conjecture.]

Conj: If \( \overline{p} : G_a \to \text{GL}_2(\mathbb{F}) \) is cus, odd, absolutely irreducible, then it is modular, i.e. \( \overline{p} \simeq \overline{p}_{f, \lambda} \) some \( f, \lambda \).