Notes on Memoryless Random Variables

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A random variable X is *memoryless* if for all numbers a and b in its range, we have

$$P(X > a + b | X > b) = P(X > a) \ . \tag{1}$$

(We are implicitly assuming that whenever a and b are both in the range of X, then so is $a+b$. The memoryless property doesn't make much sense without that assumption.) It is easy to prove that if the range of X is $[0, \infty)$, then X must be exponential. Similarly, if the range of X is $\{0, 1, 2, ...\}$, then X must be geometric.

Proof for the $[0, \infty)$ *case:* Let's define $G(a) := P(X > a)$. If X is memoryless, then G has the following properties:

1. $G(a + b) = G(a) \cdot G(b)$.
2. G is monotonically decreasing, i.e., if $a \leq b$ then $G(a) \geq G(b)$.

Property 1 follows directly from Eq. (1); Property 2 is a consequence of the definition of G. We'll see that the above two conditions imply that for all real $x \geq 0$,

$$G(x) = G(1)^x \ . \tag{2}$$

Setting $\lambda = -\log(G(1))$ would then gives $G(x) = P(X > x) = e^{-\lambda x}$, i.e., X is exponential.

To prove Eq. (1), first assume x is *rational*, i.e., $x = m/n$ for integers m and n. Using Property 1 above, we see that $G(m/n) = G(1/n)^m$. Raising both sides to the nth power yields $G(m/n)^n = (G(1/n)^m)^n = (G(1/n)^n)^m = G(1)^m$, where the last step uses Property 1 again. Taking the $1/n$th power of both sides gives $G(m/n) = G(1)^{m/n}$.

Now let x be any real number ≥ 0. Choose any two sequences q_n and r_n of rational numbers such that $q_n < x < r_n$ for all n and $q_n, r_n \to x$ as $n \to \infty$. From Property 2 above, we get

$$G(1)^{q_n} = G(q_n) \geq G(x) \geq G(r_n) = G(1)^{r_n} \ . \tag{3}$$

Letting $n \to \infty$ gives $G(1)^x = G(x)$, as desired. QED

Property 1 above is equivalent to *Cauchy's functional equation:*

$$g(a + b) = g(a) + g(b) \ ; \tag{4}$$

this is obtained from Property 1 by setting $g(x) = \log(G(x))$. The proof above shows that if g is decreasing, then the only solution of Eq. (4) is $g(x) = cx$ for some constant c. What happens if we drop the requirement that g is decreasing? Then there can be extremely strange solutions to Eq. (4)! These "pathological" solutions have some amazing properties, including 1) discontinuities at every point; 2) unbounded on every open interval; and 3) they are not "measurable." Such functions cannot be the pdf of *any* random variable!

If you're interested in learning more, start with

http://en.wikipedia.org/wiki/Cauchy_functional_equation

You might also want to check out *Lectures on Functional Equations and Their Applications* by J Aczel.